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Counting 1/8-BPS dual-giants

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Abstract: We count 1/8-BPS states in type IIB string theory on $AdS_5 \times S^5$ background which carry three independent angular momenta on S^5. These states can be counted by considering configurations of multiple dual-giant gravitons up to N in number which share at least four supersymmetries. We map this counting problem to that of counting the energy eigenstates of a system of N bosons in a 3-dimensional harmonic oscillator. We also count 1/8-BPS states with two independent non-zero spins in AdS_5 and one non-zero angular momentum on S^5 by considering configurations of arbitrary number of giant gravitons that share at least four supersymmetries.

Keywords: AdS-CFT Correspondence, D-branes.

Contents

1. Introduction

In the context of AdS/CFT correspondence [1] it is of interest to count the number of BPS states for fixed charges and supersymmetries in both $\mathcal{N} = 4$ $U(N)$ SYM and in type IIB string theory on $AdS_5 \times S^5$ background with applications to black holes in mind. The most interesting outstanding problem in this context is to count the microstates of the supersymmetric black holes in AdS_5. These black holes were first found by Gutowski and Reall [2] (and were further generalized in [3, 4]) and when lifted to 10 dimensions preserve just 2 supersymmetries [5]. This program of counting BPS states on both sides of the AdS/CFT correspondence has only been achieved so far for the simplest case of half-BPS states that carry one $U(1)$ R-charge. A generic state in string theory on $AdS_5 \times S^5$ can be specified by giving the quantum numbers $(E, S_1, S_2, J_1, J_2, J_3)$ where E is the energy in global AdS_5, S_1 and S_2 are the two independent angular momenta in AdS_5 and (J_1, J_2, J_3) are the three independent R-charges corresponding to angular momenta in S^5.

In the case of half-BPS states from the string theory point of view one can count either the multi-giant graviton states or multi-dual-giant graviton states [6]. A giant graviton is a half-BPS classical D3-brane configurations wrapping an $S^3 \subset S^5$ and rotating along one of the transverse directions to it within S^5 [7]. A dual-giant graviton is another half-BPS D3-brane configuration that wraps an $S^3 \subset AdS_5$ and rotating along a maximal circle of S^5 [8, 9]. From the point of view of giants the stringy exclusion principle manifests itself in the fact that the maximum angular momentum that a single giant graviton can carry is N. The same stringy exclusion principle from the point of view of dual-giant gravitons is that there is an upper limit on the number of dual-giants which is again given by N. From the perspective of supergravity and probe branes this upper limit on the number of dual-giants has to do with the way the 5-form RR-flux decreases inside each dual-giant by one unit (see for instance [10, 6]). To count the half-BPS states one can consider an arbitrary number of half-BPS giants, treated as bosons, in an N-level equally spaced spectrum. The same half-BPS state counting can be done by counting configurations of multiple dual-giant gravitons, treated as bosons, in an infinite equally spaced spectrum with an upper limit on the dual-giants given by N.

Progress in counting BPS states with lower supersymmetry in the SYM has been made difficult by the fact that the number of these states is known to jump from zero YM coupling to non-zero coupling. In this note we aim to count a subclass of 1/8-BPS states in type IIB string theory on $AdS_5 \times S^5$ geometry that carry three independent $U(1)$ R-charges J_1, J_2 and J_3 and have the energy (conjugate to the global AdS_5 time) $E = J_1 + J_2 + J_3$. This subclass of 1/8-BPS states preserve the $SO(4)$ invariance of the $S^3 \subset AdS_5$. We will do the counting of 1/8-BPS states by considering all possible multiple dual-giant graviton configurations which preserve a common set of at least 4 supersymmetries. Since we are interested in putting together half-BPS dual-giants to obtain the 1/8-BPS states the argument that there should be an upper limit on the number of dual-giants given by N continues to be valid.

In [11] Mikhailov described the most general 1/8-BPS giant gravitons as the intersections of holomorphic complex surfaces in $\mathbb{C}^3$ with S^5. So for counting the 1/8-BPS states one may try to quantize these giant graviton configurations and count them. We have been informed that this has been recently achieved by Biswas, Gaiotto, Lahiri and Minwalla [12]. Our results agree with theirs. The agreement for 1/8-BPS states suggests that both Mikhailov's giants and our dual-giants present dual descriptions of the same set of states like in the half-BPS case.

We also consider 1/8-BPS states which carry non-zero (S_1, S_2, J_1) and have $E = S_1 + S_2 + J_1$. This time we consider multiple giant graviton configurations which have angular momenta in AdS_5 part as well that share at least 4 supersymmetries.

The rest of this note is organized as follows. In section 2 we construct a class of dual-giant solutions which share at least 4 supersymmetries. We show that this class of solutions is the full set of 1/2-BPS dual-giants that share a given 4 supersymmetries with the rest in their class. We find the solution space and a sympletic form on this space and quantize it. In section 3 we give the partition function of the 1/8-BPS states with (J_1, J_2, J_3) charges. In section 4 we consider the problem of counting 1/8-BPS states with (S_1, S_2, J_1) charges. In section 5 we conclude with a summary and some remarks.

2. 1/8-BPS dual-giant configurations

In this section we will find the most general dual-giant configurations which preserve a given 4 supersymmetries in $AdS_5 \times S^5$. As mentioned in the introduction we put half-BPS dual-giants together to make the lower supersymmetric configurations. A given half-BPS dual-giant has momentum along one of the maximal circles of S^5. A general dual-giant configuration contains dual-giants rotating along different maximal circles. There is no a priori guarantee that such a generic configuration preserves any supersymmetries. Below we will look for a class of single dual-giant solutions which shares at least 4 supersymmetries with the first one.

2.1 The dual-giant solutions

We regard S^5 as a submanifold in $\mathbb{C}^3$ with coordinates

$$(z_1 = l\mu_1\, e^{i\,\xi_1}, z_2 = l\mu_2\, e^{i\,\xi_2}, z_3 = l\,\mu_3\, e^{i\,\xi_3}) \tag{2.1}$$

as $z_i\bar{z}_i = l^2$ and $(\mu_1, \mu_2, \mu_3) = (\sin\alpha, \cos\alpha\sin\beta, \cos\alpha\cos\beta)$:

$$ds^2|_{S^5} = l^2\left(d\alpha^2 + \cos^2\alpha\beta^2 + \mu_i^2 d\xi_i^2\right) = dz_i d\bar{z}_i \tag{2.2}$$

The vielbeins are

$$\begin{array}{lll} e^0 = V^{1/2}(r)\, dt, & e^1 = V^{-1/2}(r)\, dr, & e^2 = r\, d\theta, \\ e^3 = r\,\nu_1\, d\phi_1, & e^4 = r\,\nu_2\, d\phi_2, & e^5 = l\, d\alpha, \\ e^6 = l\, \cos\alpha\, d\beta, & e^7 = l\,\mu_1\, d\xi_1, & e^8 = l\,\mu_2\, d\xi_2, \\ e^9 = l\,\mu_3\, d\xi_3. & & \end{array} \tag{2.3}$$

where $\nu_1 = \cos\theta$, $\nu_2 = \sin\theta, V(r) = 1 + r^2/l^2$. The five form RR field strength is

$$F^{(5)} = -\frac{4}{l}\left[e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^5 \wedge e^6 \wedge e^7 \wedge e^8 \wedge e^9\right]. \tag{2.4}$$

The 4-form potential can be written as

$$C^{(4)} = r^4\, \cos\theta\, \sin\theta\, d(t/l) \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 + l^4\, \cos^4\alpha\, \sin\beta\, \cos\beta\, d\beta \wedge d\xi_1 \wedge d\xi_2 \wedge d\xi_3. \tag{2.5}$$

The general embedding of a D3-brane wrapped on the $S^3 \in AdS_5$ is given by

$$\begin{array}{lllll} t = \sigma_0 = \tau, & r = r(\tau), & \theta = \sigma_1, & \phi_1 = \sigma_2, & \phi_2 = \sigma_3, \\ \alpha = \alpha(\tau), & \beta = \beta(\tau), & \xi_1 = \xi_1(\tau), & \xi_2 = \xi_2(\tau), & \xi_3 = \xi_3(\tau). \end{array} \tag{2.6}$$

The pull-back of $C^{(4)}$ onto the world-volume is $C^{(4)}_{\sigma_0\sigma_1\sigma_2\sigma_3} = l^{-1}r^4\, \cos\theta\, \sin\theta$. The DBI and WZ lagrangian density is

$$\mathcal{L} = -\frac{N}{2\,\pi^2\, l^4} r^3 \cos\sigma_1 \sin\sigma_1 \left[\Delta^{1/2} - \frac{r}{l}\right] \tag{2.7}$$

where

$$\Delta = V(r) - \frac{\dot{r}^2}{V(r)} - l^2(\dot{\alpha}^2 + \cos^2\alpha\, \dot{\beta}^2 + \mu_1^2\, \dot{\xi}_1^2 + \mu_2^2\, \dot{\xi}_2^2 + \mu_3^2\, \dot{\xi}_3^2). \tag{2.8}$$

We can integrate over the angles σ_i to get Action $= \int dt\, L$ where the effective point-particle Lagrangian L is

$$L = \int d\sigma_1\, d\sigma_2\, d\sigma_3\ \mathcal{L} = -\frac{N}{l^4} r^3 \left[\Delta^{1/2} - \frac{r}{l}\right] \tag{2.9}$$

The conjugate variables of the classical mechanics model are

$$\begin{aligned} P_r &= \frac{N\, r^3\, \dot{r}}{l^4\, V(r)\, \Delta^{1/2}}, & P_\alpha &= \frac{N\, r^3\, \dot{\alpha}}{l^2\, \Delta^{1/2}}, & P_\beta &= \frac{N\, r^3\, \cos^2\alpha\, \dot{\beta}}{l^2\, \Delta^{1/2}}, \\ P_{\xi_1} &= \frac{N\, r^3\, \mu_1^2\, \dot{\xi}_1}{l^2 \Delta^{1/2}}, & P_{\xi_2} &= \frac{N\, r^3\, \mu_2^2\, \dot{\xi}_2}{l^2 \Delta^{1/2}}, & P_{\xi_3} &= \frac{N\, r^3\, \mu_3^2\, \dot{\xi}_3}{l^2 \Delta^{1/2}} \end{aligned} \tag{2.10}$$

By looking at the equations of motion of the lagrangian in (2.9) it is easy to see that setting

$$\dot{r} = \dot{\alpha} = \dot{\beta} = 0, \quad |\dot{\xi}_1| = |\dot{\xi}_2| = |\dot{\xi}_3| = 1/l \tag{2.11}$$

solves them.

2.2 Kappa projections

In this subsection we will show that the solutions in (2.11) are all supersymmetric. We will see that depending on the signs of $\dot{\xi}_i$ there are 8 disjoint sets of these supersymmetric dual-giant solutions which do not share any common supersymmetries. Different dual-giant solutions corresponding to different values of r, α, β and $\xi_i(\tau = 0)$ but with fixed signs for $\dot{\xi}_i$ preserve at least 4 common supersymmetries of the background $AdS_5 \times S^5$ geometry.

To write down the kappa projection equation for the probe D3-brane we need the world-volume gamma matrices, which are

$$\begin{aligned} \gamma_\tau &= V^{1/2}(r)\, \Gamma_0 + \frac{\dot{r}}{V^{1/2}(r)}\, \Gamma_1 + l\, (\dot{\alpha}\, \Gamma_5 + \cos\alpha\, \dot{\beta}\, \Gamma_6 + \sum_{i=1}^{3} \dot{\xi}_i\, \mu_i\, \Gamma_{6+i}) \\ \gamma_{\sigma_1} &= r\, \Gamma_2, \quad \gamma_{\sigma_2} = r\, \cos\sigma_1\, \Gamma_3, \quad \gamma_{\sigma_3} = r\, \sin\sigma_1\, \Gamma_4 \end{aligned} \tag{2.12}$$

where Γ_a are the 10-dimensional tangent space gamma matrices satisfying $\{\Gamma_a, \Gamma_b\} = 2\, \eta_{ab}$. The world-volume gamma matrices of (2.2) satisfy $\{\gamma_m, \gamma_n\} = 2\, g_{mn}$. The kappa projection matrix is:

$$\begin{aligned} \Gamma &= \frac{1}{4! \sqrt{-\det\, g_{mn}}}\, \epsilon^{mnpq} \gamma_{mnpq} \\ &= \Delta^{-1/2} \left[V^{1/2}(r)\, \Gamma_0 + \frac{\dot{r}}{V^{1/2}(r)}\, \Gamma_1 + l\, (\dot{\alpha} \Gamma_5 + \cos\alpha\, \dot{\beta}\, \Gamma_6 + \sum_{i=1}^{3} \dot{\xi}_i\, \mu_i\, \Gamma_{6+i}) \right] \Gamma_{234} \end{aligned} \tag{2.13}$$

where Δ is defined as before. With this the kappa projection on the Killing spinor ϵ of the background $AdS_5 \times S^5$ geometry is

$$\Gamma\, \epsilon = i\, \epsilon. \tag{2.14}$$

The chirality convention for ϵ is

$$\Gamma_0 \cdots \Gamma_9 \epsilon = -\epsilon. \tag{2.15}$$

The Killing spinor equations of $AdS_5 \times S^5$ are:

$$D_\mu \epsilon + \frac{i}{1920} F^{(5)}_{\nu_1\nu_2\nu_3\nu_4\nu_5} \Gamma^{\nu_1\nu_2\nu_3\nu_4\nu_5} \Gamma_\mu \, \epsilon = 0 \tag{2.16}$$

The solution of these equations can be written as:

$$\begin{aligned}\epsilon = \; & e^{\frac{i}{2}\alpha\,\Gamma_5\,\tilde\gamma}\, e^{\frac{i}{2}\beta\,\Gamma_6\tilde\gamma}\, e^{\frac{1}{2}\xi_1\Gamma_{57}}\, e^{\frac{1}{2}\xi_2\Gamma_{68}}\, e^{\frac{i}{2}\xi_3\Gamma_9\,\tilde\gamma} \\ & \times e^{\frac{i}{2}\sinh^{-1}(\frac{r}{l})\,\Gamma_1\,\gamma}\, e^{\frac{i}{2l}t\,\Gamma_0\,\gamma}\, e^{\frac{1}{2}\theta\,\Gamma_{12}}\, e^{\frac{1}{2}\phi_1\Gamma_{13}}\, e^{\frac{1}{2}\phi_2\Gamma_{24}}\, \epsilon_0 \equiv M\,\epsilon_0.\end{aligned} \tag{2.17}$$

where $\gamma = \Gamma^{01234}$ and $\tilde\gamma = \Gamma^{56789}$. The full kappa projection equation then reads

$$\left[V^{1/2}(r)\,\Gamma_0 - i\,\Delta^{1/2}\,\Gamma_{234} + \frac{\dot r}{V^{1/2}(r)}\,\Gamma_1 + l\,(\dot\alpha\,\Gamma_5 + \cos\alpha\,\dot\beta\,\Gamma_6 + \sum_{i=1}^{3}\dot\xi_i\,\mu_i\,\Gamma_{6+i})\right] M\epsilon_0 = 0 \tag{2.18}$$

We now show that the solutions in (2.11) are supersymmetric. Let us first choose the signs of ξ_i's to be the same and positive. On the solution the kappa projection equation becomes

$$\left[V^{1/2}(r_0)\,\Gamma_0 - i\frac{r_0}{l}\Gamma_{234} + \mu_1\,\Gamma_7 + \mu_2\,\Gamma_8 + \mu_3\,\Gamma_9\right] M\epsilon_0 = 0 \tag{2.19}$$

where r_0 is the value of r in the solutions. Some useful intermediate steps in simplifying this equation are:

$$\begin{aligned}
\Gamma_0 M \;&=\; M\left[V^{1/2}\Gamma_0 - i\frac{r_0}{l}e^{-i\frac{t}{l}\Gamma_0\gamma}\left(\cos\theta\,e^{-\phi_1\Gamma_{13}}\Gamma_1 + \sin\theta\,e^{-\phi_2\Gamma_{24}}\Gamma_2\right)\Gamma_0\,\gamma\right],\\
\Gamma_2 M \;&=\; M\left[V^{1/2}e^{-i\frac{t}{l}\Gamma_0\gamma}\left(\cos\theta\,e^{-\phi_2\Gamma_{24}}\Gamma_2 - \sin\theta\,e^{-\phi_1\Gamma_{13}}\Gamma_1\right) - i\frac{r}{l}e^{-\phi_1\Gamma_{13}-\phi_2\Gamma_{24}}\Gamma_{12}\gamma\right],\\
\Gamma_3 M \;&=\; M\left[V^{1/2}\,e^{-\phi_1\Gamma_{13}}\,e^{-i\frac{t}{l}\Gamma_0\gamma}\Gamma_3 - i\frac{r_0}{l}\left(\cos\theta\,\Gamma_1 + \sin\theta\,e^{-(\phi_1\Gamma_{13}+\phi_2\Gamma_{24})}\Gamma_2\right)\Gamma_3\gamma\right],\\
\Gamma_4 M \;&=\; M\left[V^{1/2}\,e^{-\phi_2\Gamma_{24}}\,e^{-i\frac{t}{l}\Gamma_0\gamma}\Gamma_4 - i\frac{r_0}{l}\left(\cos\theta\,e^{-(\phi_1\Gamma_{13}+\phi_2\Gamma_{24})}\Gamma_1 + \sin\theta\,\Gamma_2\right)\Gamma_4\gamma\right],\\
\Gamma_7 M \;&=\; M\Big[\cos\alpha\,\cos\beta\,e^{-\xi_1\Gamma_{57}}\,e^{-i\xi_3\Gamma_9\tilde\gamma}\Gamma_7 - i\cos\alpha\,\sin\beta\,e^{-\xi_1\Gamma_{57}}e^{-\xi_2\Gamma_{68}}\Gamma_{67}\tilde\gamma - i\sin\alpha\,\Gamma_{57}\tilde\gamma\Big],\\
\Gamma_8 M \;&=\; M\Big[\cos\alpha\,\cos\beta\,e^{-i\xi_3\Gamma_9\tilde\gamma}e^{-\xi_2\Gamma_{68}}\Gamma_8 - i\cos\alpha\,\sin\beta\,\Gamma_{68}\tilde\gamma - i\sin\alpha\,e^{-\xi_2\Gamma_{68}}e^{-\xi_1\Gamma_{57}}\Gamma_{58}\tilde\gamma\Big],\\
\Gamma_9 M \;&=\; M\Big[\cos\alpha\,\cos\beta\,\Gamma_9 - i\cos\alpha\,\sin\beta\,e^{-i\xi_3\Gamma_9\tilde\gamma}e^{-\xi_2\Gamma_{68}}\Gamma_{69}\tilde\gamma - i\sin\alpha\,e^{-i\xi_3\Gamma_9\tilde\gamma}e^{-\xi_1\Gamma_{57}}\Gamma_{59}\tilde\gamma\Big].
\end{aligned}$$

Let us also register the following identity:

$$\begin{aligned}\Gamma_{234}M \;&=\; M\left[V^{1/2}e^{-i\frac{t}{l}\Gamma_0\gamma}\left(\cos\theta\,e^{-\phi_1\Gamma_{13}}\Gamma_{234} - \sin\theta e^{-\phi_2\Gamma_{24}}\Gamma_{134}\right) - i\frac{r}{l}\Gamma_{1234}\gamma\right]\\ &=\; -M\left[V^{1/2}e^{-i\frac{t}{l}\Gamma_0\gamma}\left(\cos\theta\,e^{-\phi_1\Gamma_{13}}\Gamma_1 + \sin\theta e^{-\phi_2\Gamma_{24}}\Gamma_2\right)\Gamma_0\gamma + i\frac{r}{l}\Gamma_0\right]\end{aligned} \tag{2.20}$$

so that we have

$$\left[V^{1/2}\Gamma_0 - i\frac{r}{l}\Gamma_{234}\right]\epsilon = M\Gamma_0\epsilon_0. \tag{2.21}$$

Using these identities eq. (2.19) can be rewritten as

$$\begin{aligned}M\Big[\Gamma_0 - i\mu_1\,\mu_2\,e^{-\xi_1\Gamma_{57}-\xi_2\Gamma_{68}}\tilde\gamma(\Gamma_{67}+\Gamma_{58}) + \mu_1\,\mu_3\,e^{-\xi_1\Gamma_{57}}\,e^{-i\xi_3\Gamma_9\tilde\gamma}(\Gamma_7 - i\Gamma_{59}\tilde\gamma)\\ +\mu_2\,\mu_3\,e^{-i\xi_3\Gamma_9\tilde\gamma}\,e^{-\xi_2\,\Gamma_{68}}\,(\Gamma_8 - i\,\Gamma_{69}\,\tilde\gamma) - i\tilde\gamma\,(\mu_1^2\,\Gamma_{57} + \mu_2^2\,\Gamma_{68}) + \mu_3^2\,\Gamma_9\Big]\epsilon_0 = 0.\end{aligned} \tag{2.22}$$

This equation can be solved by imposing the following projections on ϵ_0

$$\begin{aligned}(\Gamma_{67}+\Gamma_{58})\epsilon_0 &= 0, \qquad (\Gamma_7 - i\,\Gamma_{59}\,\tilde\gamma)\epsilon_0 = 0, \qquad (\Gamma_8 - i\,\Gamma_{69}\,\tilde\gamma)\epsilon_0 = 0,\\ (\Gamma_0 - i\Gamma_{57}\tilde\gamma)\epsilon_0 &= 0, \qquad (\Gamma_9 + i\,\Gamma_{57}\,\tilde\gamma)\epsilon_0 = 0.\end{aligned} \tag{2.23}$$

It is easy to see that out of these the independent projections are (as the second and the fourth projections are equivalent and so are the third and the fifth)

$$(\Gamma_0 - i\Gamma_{57}\tilde\gamma)\epsilon_0 = 0, \quad (\Gamma_0 - i\Gamma_{68}\tilde\gamma)\epsilon_0 = 0, \quad (\Gamma_0 + \Gamma_9)\epsilon_0 = 0. \tag{2.24}$$

By considering the special cases of setting two of $\dot\xi_i$ to zero and finding the kappa projections one recognizes these three projection conditions as the ones for half-BPS dual-giants which rotate in z_1, z_2, z_3 planes respectively. One can further see that different signs for $\dot\xi_i$ result in similar equations with different relative signs in eqs. (2.24) corresponding to reversing the direction of motion of some of the three half-BPS dual-giants in their respective planes. That is, for the solution with

$$(l\,\dot\xi_1, l\,\dot\xi_2, l\,\dot\xi_3) = (\lambda_1,\lambda_2,\lambda_3), \quad \lambda_i = \pm 1 \tag{2.25}$$

we have

$$(\Gamma_0 - i\,\lambda_1\,\Gamma_{57}\tilde\gamma)\epsilon_0 = 0, \quad (\Gamma_0 - i\,\lambda_2\,\Gamma_{68}\tilde\gamma)\epsilon_0 = 0, \quad (\Gamma_0 + \ \lambda_3\,\Gamma_9)\epsilon_0 = 0. \tag{2.26}$$

Thus we have seen that the solutions in (2.11) with all signs of $\dot\xi_i$ being positive preserve at least 4 supersymmetries consistent with the projections in (2.24) for any arbitrary values of α, β and r. However one should note that for fixed values of α, β and r the projection equation (2.22) can give 16 supersymmetries. This can be seen by redefining the tangent space Γ-matrices appropriately.

In the next subsection we will prove the converse, namely that these are <u>all</u> the dual-giant solutions which preserve at least these four given supersymmetries.

2.3 Necessity of the conditions (2.11)

In this subsection, unlike the previous one, we will not assume the solutions (2.11) to begin with. We will rather show that these solutions (with signs (2.25)) follow uniquely if we demand the kappa projections (2.14),(2.13) on the 1/8-th supersymmetric subspace of spinors ϵ defined by (2.17),(2.26). This will therefore show that those solutions are the full set of solutions consistent with the specified set of 4 supersymmetries.

To begin, we rewrite the supersymmetry projections (2.24) (we will assume all λ_i's to be $+1$, the generalization to arbitrary signs being straightforward)

$$(\Gamma_0 - i\Gamma_{57}\tilde\gamma)\,\epsilon_0 = 0, \quad (\Gamma_0 - i\Gamma_{68}\tilde\gamma)\,\epsilon_0 = 0, \quad (\Gamma_0 + \Gamma_9)\,\epsilon_0 = 0. \tag{2.27}$$

We can use these to simplify the full expression (2.17) of the Killing spinor on $AdS_5 \times S^5$ to

$$\begin{aligned}\epsilon \,=\, &\times e^{\frac{i}{2}\sinh^{-1}(\frac{r}{l})\,\Gamma_1\,\gamma}\, e^{\frac{i}{2}(\frac{t}{l}+\xi_1+\xi_2+\xi_3)\,\Gamma_0\,\gamma}\, e^{\frac{1}{2}\theta\,\Gamma_{12}}\, e^{\frac{1}{2}\phi_1\Gamma_{13}}\, e^{\frac{1}{2}\phi_2\Gamma_{24}}\\ &e^{\frac{i}{2}\alpha\,\Gamma_5\,\tilde\gamma}\, e^{\frac{i}{2}\beta\,\Gamma_6\tilde\gamma}\,\epsilon_0 \,\equiv\, \hat M\,\epsilon_0.\end{aligned} \tag{2.28}$$

To simplify the kappa projection equations (2.18) we will need

$$\begin{aligned}
\Gamma_1\,\hat{M} &= \hat{M}\,e^{-i\frac{t}{l}\Gamma_0\gamma}[\cos\theta\,e^{-\phi_1\,\Gamma_{13}}\Gamma_1+\sin\theta\,e^{-\phi_2\,\Gamma_{24}}\,\Gamma_2],\\
\Gamma_5\,\hat{M} &= \hat{M}\,[\cos\beta\,\Gamma_5+i\,\sin\beta\,\Gamma_{56}\,\tilde{\gamma}]\\
\Gamma_6\,\hat{M} &= \hat{M}\,[\cos\alpha\,\Gamma_6-i\,\sin\alpha\,\cos\beta\,\Gamma_{56}\tilde{\gamma}+\sin\alpha\,\sin\beta\,\Gamma_5],\\
\Gamma_7\,\hat{M} &= \hat{M}\,[\mu_3\,\Gamma_7-i\,\mu_2\,\Gamma_{67}\,\tilde{\gamma}-i\,\mu_1\,\Gamma_{57}\,\tilde{\gamma}]\\
\Gamma_8\,\hat{M} &= \hat{M}\,[\mu_3\,\Gamma_8-i\,\mu_2\,\Gamma_{68}\,\tilde{\gamma}-i\,\mu_1\,\Gamma_{58}\,\tilde{\gamma}]\\
\Gamma_9\,\hat{M} &= \hat{M}\,[\mu_3\,\Gamma_9-i\,\mu_2\,\Gamma_{69}\,\tilde{\gamma}-i\,\mu_1\,\Gamma_{59}\,\tilde{\gamma}]
\end{aligned}\tag{2.29}$$

along with $\Gamma_0\,M$ and $\Gamma_{234}\,M$ relations from the earlier section. Since we want to demand the projections in (2.27) to be valid at every point of the world-volume of the D3-brane and so for convenience we can set $\tau=0$ and $\sigma_1=\sigma_2=0$. Then the first three terms in eq. (2.18) can be simplified to

$$[V^{1/2}\Gamma_0+\frac{\dot{r}}{V^{1/2}}\Gamma_1-i\Delta^{1/2}\Gamma_{234}]\hat{M}=\hat{M}[(V-\frac{r}{l}\Delta^{1/2})\Gamma_0+\frac{\dot{r}}{V^{1/2}}+i\Gamma_{01}\gamma V^{1/2}(\frac{r}{l}-\Delta^{1/2})]\tag{2.30}$$

The last five terms in eq. (2.18) can be simplified to

$$\begin{aligned}
&[\dot{\alpha}\,\Gamma_5+\cos\alpha\,\dot{\beta}\,\Gamma_6+\sum_{i=1}^{3}\mu_i\,\dot{\xi}_i\,\Gamma_{6+i}]\,\hat{M}\;=\hat{M}\,[(\cos\beta\,\dot{\alpha}+\sin\alpha\,\cos\alpha\,\sin\beta\,\dot{\beta})\,\Gamma_5\\
&\quad+\Gamma_6\,\cos^2\alpha\,\dot{\beta}+i\,\Gamma_{56}\,\tilde{\gamma}\,(\sin\beta\,\dot{\alpha}-\sin\alpha\,\cos\alpha\,\cos\beta\,\dot{\beta})+i\,\Gamma_{59}\,\tilde{\gamma}\,\mu_1\,\mu_2\,(\dot{\xi}_1-\dot{\xi}_3)\\
&\quad-i\,\Gamma_{58}\,\tilde{\gamma}\,\mu_1\mu_2\,(\dot{\xi}_2-\dot{\xi}_1)-i\,\Gamma_{69}\,\tilde{\gamma}\,(\dot{\xi}_2-\dot{\xi}_3)-i\,\mu_1^2\,\dot{\xi}_1\,\Gamma_{57}\tilde{\gamma}-i\,\mu_2^2\,\dot{\xi}_2\,\Gamma_{68}\,\tilde{\gamma}+\mu_3^2\,\dot{\xi}_3\,\Gamma_9]
\end{aligned}\tag{2.31}$$

Thus the full kappa projection equation (2.18) is a linear combination of various products of 10-dimensional γ-matrices for the tangent space acting on a constant spinor ϵ_0 which is arbitrary but for the projection equations (2.27). Recall that a linearly independent basis of these 32×32 matrices is given by $\Gamma_{m\,n\cdots}$'s. Since ϵ_0 is arbitrary only upto the projections in eq. (2.27) we have to use them to eliminate operators which kill ϵ_0 and then put the coefficients of the remaining independent $\Gamma_{m\,n\cdots}$ to zero. Doing this we see that $\dot{\beta}=0$ as the coefficient of Γ_6 (which does not appear anywhere else), which then implies $\dot{\alpha}=0$. Next notice that Γ_{58}, Γ_{59} and Γ_{69} would not occur anywhere else and so we have to set $\dot{\xi}_1=\dot{\xi}_2=\dot{\xi}_3\equiv\omega/l$. Note that these are not absolute values - unlike in the analysis of equations of motion. Since $\dot{r}$ appears with Γ_1 and Γ_1 does not occur anywhere else $\dot{r}=0$. The coefficient of $\Gamma_{01}\gamma$ has to be set to zero too and this implies $\omega=\pm 1$. Finally using the projection equations (2.27) we can convert $\Gamma_{57}\tilde{\gamma}$, $\Gamma_{68}\,\tilde{\gamma}$ and Γ_9 into Γ_0. Using the relations obtained so far and then setting the coefficient of Γ_0 to zero requires $\omega=1$. That completes the proof that the set of solutions we found is complete.

2.4 Reduced phase space

The supersymmetry constraints (2.27) translate to the following in terms of the canonical momenta (see (2.10))

$$\begin{aligned}
P_r &= 0, & P_\alpha &= 0, & P_\beta &= 0,\\
P_{\xi_1}-\tfrac{N}{l^2}r^2\,\mu_1^2 &= 0, & P_{\xi_2}-\tfrac{N}{l^2}r^2\,\mu_2^2 &= 0, & P_{\xi_3}-\tfrac{N}{l^2}r^2\,\mu_3^2 &= 0
\end{aligned}\tag{2.32}$$

Thus, the original 12-dimensional phase space is reduced to a six-dimensional one. We will see below that the reduced phase space can be described entirely by the coordinates $r, \alpha, \beta, \xi_1, \xi_2, \xi_3$ or by the complex coordinates (2.34). The symplectic structure on the reduced phase space can be derived by the following Dirac brackets:

$$\begin{aligned} \{q_i, q_j\}_{DB} &= \{q_i, q_j\}_{PB} - \{q_i, f_a\}_{PB}\ M^{-1}_{ab}\ \{f_b, q_j\}_{PB} \\ M_{ab} &= \{f_a, f_b\}_{PB} \end{aligned} \tag{2.33}$$

where q_i refers to one of the six coordinates $r, \alpha, \beta, \xi_1, \xi_2, \xi_3$ and f_a refers to one of the six constraints (2.32).

The calculation of the Dirac brackets on the supersymmetric subspace is a straightforward generalization of the half-BPS giant [13] and dual-giant [14] gravitons. To proceed with the calculations, let us define the complex coordinates

$$\zeta_i = r_i e^{i\xi_i}, \qquad (r_1, r_2, r_3) \equiv r(\sin\alpha, \cos\alpha\sin\beta, \cos\alpha\cos\beta) = r(\mu_1, \mu_2, \mu_3), \tag{2.34}$$

In terms of the coordinates r_i, ξ_i and their conjugate momenta, the constraints become

$$f_i \equiv P_{r_i} = 0, \qquad f_{3+i} \equiv P_{\xi_i} - \frac{N}{l^2} r_i^2 = 0, \qquad i = 1, 2, 3 \tag{2.35}$$

The non-zero elements of the matrix M_{ab} are given by

$$\{f_i, f_{3+j}\}_{PB} = \left\{P_{r_i}, -\frac{N}{l^2} r_j^2\right\}_{PB} = \frac{2N}{l^2} r_j \delta_{ij}$$

Thus, for example,

$$\{\xi_1, r_1\}_{DB} = -\{\xi_1, f_4\}_{PB}\, \frac{l^2}{2Nr_1}\, \{f_1, r_1\}_{PB} = \frac{l^2}{2Nr_1} \tag{2.36}$$

The Dirac brackets are summarized by

$$\{\xi_i, r_j^2\}_{DB} = \frac{l^2}{N}\delta_{ij} \implies \{\zeta_i, \bar{\zeta}_j\}_{DB} = -i\frac{l^2}{N}\delta_{ij} \tag{2.37}$$

Symplectic structure: The conclusion of this subsection is that the reduced phase space is simply $\mathbb{C}^3$ with the symplectic form

$$\omega = i\frac{N}{l^2}\, d\zeta_i \wedge d\bar{\zeta}_i \tag{2.38}$$

2.5 Hamiltonian and charges

Since translations along ξ_i are symmetries of the D3-brane action the momenta P_{ξ_i} are conserved charges. The canonical hamiltonian, with $P_\alpha = P_\beta = P_r = 0$, becomes

$$H = \frac{1}{l}\sqrt{\left(\sqrt{\sum_{i=1}^{3}\frac{P_{\xi_i}^2}{\mu_i^2}} + \frac{Nr^4}{l^4}\right)^2 + \frac{r^2}{l^2}\left(\sqrt{\sum_{i=1}^{3}\frac{P_{\xi_i}^2}{\mu_i^2}} - \frac{Nr^2}{l^2}\right)^2} - \frac{Nr^4}{l^5} \tag{2.39}$$

After imposing the remaining three constraints from (2.32) it reduces to

$$H = \frac{Nr^2}{l^3} = \frac{N}{l^3}\zeta_i\bar{\zeta}_i = \frac{1}{l}(P_{\xi_1} + P_{\xi_2} + P_{\xi_3}) \tag{2.40}$$

which is simply the Hamiltonian of a 3-dimensional simple harmonic oscillator.

As expected, the BPS constraint equations automatically satisfy the equations of motion. The solutions to (2.32) and (2.40) are

$$\begin{aligned} r(t) &= r_0, \\ z_k(t) &= l\,\mu_k^{(0)}\, e^{i(\xi_k^{(0)} + \frac{t}{l})} \end{aligned} \tag{2.41}$$

for $k = 1$, 2 and 3. The motion is obviously periodic with period $\Delta t = 2\pi l$. Thus we have a 6-dimensional space of solutions parametrized by: $(\mu_k^{(0)}, \xi_k^{(0)}, r_0)$. It is well-known that the space of solutions of a dynamical system (modulo time-evolution) can be identified with its phase space. Folowing section 2.4, this six-dimensional space of solutions, viewed as a symplectic space, is $\mathbb{C}^3$. We will show in the next subsection that the motion in S^5 (second line of (2.41)) are all in maximal circles in S^5 which are related to the one in z_1-plane by $U(3)$ rotations.

2.6 Interpretation of the solutions as maximal circles

Let us now try to understand the full solution set of dual-giants which preserve at least 4 common supersymmetries. A maximal circle on S^5 can be parametrized by 8 parameters. To see this note that every maximal circle can be obtained by intersecting S^5 with an $\mathbb{R}^2$ passing through the origin in $\mathbb{R}^6$ in which S^5 is embedded. The space of these planes has a dimension of $\frac{SO(6)}{SO(4)\times SO(2)}$ which is 8. Let us consider a subspace of maximal circles parametrized as $\vec{z}(\theta) = (z_1(\theta), z_2(\theta), z_3(\theta)) = e^{i\,\theta}\vec{z}(0)$, $\theta \in (0, 2\pi]$ where z_i are complex coordinates in $\mathbb{C}^3$ in which we embed the S^5 by $|z_1|^2 + |z_2|^2 + |z_3|^2 = l^2$. Any such circle can be obtained by a $U(3)$ rotation (defined upto a $U(2)$) on a reference circle, for example,

$$\vec{z}(\theta) \equiv U e^{i\theta}(l, 0, 0), \tag{2.42}$$

where we can take U to be

$$U = \begin{pmatrix} e^{i\xi_1}\sin\alpha & -e^{i\xi_1}\cos\alpha & 0 \\ e^{i\xi_2}\cos\alpha\sin\beta & e^{i\xi_2}\sin\alpha\sin\beta & e^{i\xi_2}\cos\beta \\ e^{i\xi_3}\cos\alpha\cos\beta & e^{i\xi_3}\sin\alpha\cos\beta & -e^{i\xi_3}\sin\beta \end{pmatrix} V. \tag{2.43}$$

Here $V \in U(2)$ is an arbitrary matrix that leaves the column vector $(l, 0, 0)$ invariant. Therefore the space of these circles can be identified with $\frac{U(3)}{U(2)}$. This space has five real dimensions and gives the parameterization (2.1) of the vector $\vec{z} = (l, 0, 0)$. We can choose the representation (2.1) for z_i. The time-dependence of a half-BPS dual-giant on the circle (2.42) is given by putting $\theta = t/l$. Hence after the $U(3)$-rotation (2.43), the motion on the generic maximal circle coincides with the generic S^5-motion of 1/8-BPS dual-giants (see second line of (2.41)). We conclude that every dual-giant (of a given size r_0) in the full set of 1/8-BPS dual-giant solutions is simply related to every other by a $U(3)$ rotation.

JHEP03(2007)031

3. Counting BPS states with (J_1, J_2, J_3)

3.1 $\frac{1}{8}$-BPS states

We found in section 2.4 that classically the reduced single-particle phase space is simply $\mathbb{C}^3$, with the symplectic form (2.38) and a 3-dimensional simple harmonic oscillator Hamiltonian, (2.40).

Since the semiclassical quantization of a simple harmonic oscillator is exact, quantum mechanically the single-particle Hilbert space is given by 3-dimensional simple harmonic oscillator eigenstates, viz.

$$|n_1, n_2, n_3\rangle = \prod_{i=1,2,3} \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}} |0\rangle \tag{3.1}$$

The hamiltonian and the charges are given by

$$\begin{aligned} P_{\xi_i} &\equiv J_i = n_i, \qquad i = 1, 2, 3 \\ lH &= n_1 + n_2 + n_3 \end{aligned} \tag{3.2}$$

Here the classical phase space variables ζ_i, $\bar{\zeta}_j$ are quantized as $(l/\sqrt{N})\ \zeta_i \to a_i$, $(l/\sqrt{N})\ \bar{\zeta}_j \to a_j^\dagger$. The operator ordering of the charges has been obtained by generalizing from the half-BPS case [15]; in the Hamiltonian we have dropped the zero point energy which is not important for our purposes. We note that the conserved momenta $(P_{\xi_1}, P_{\xi_2}, P_{\xi_3})$ by construction correspond to the three angular momenta (J_1, J_2, J_3) of $SO(6)$. In what follows we will use the notation J_i in stead of P_{ξ_i}.

As we have argued earlier, putting any number of particles in this reduced Hilbert space is consistent with 1/8 supersymmetry. Since they are BPS objects (with respect to each other) the total energy is given by the sum of the individual energies (like in the case of half-BPS states). Further we can have more than one dual-giants with exactly the same quantum number and so they can be treated as bosonic objects. Since each dual-giant being an S^3 inside AdS_5 acts as a domain wall and therefore considerations of [10, 6] apply. This restricts the maximum number of dual-giants to N. The total angular momenta should also be given by the sum of those for the individual dual-giants:

$$J_i = \sum_{k=1}^{N} J_i^{(k)} \quad \text{for} \quad i = 1,\ 2,\ 3. \tag{3.3}$$

From (3.2) it is easy to see that the partition function for the dual-giant graviton system, consistent with 1/8 supersymmetries, is given by that for N bosons in a 3-dimensional simple harmonic oscillator. Here we can identify these bosons with dual-giant gravitons. By including the $J_i = 0$ state we may simply count all states with a total of N bosons with some of them sitting at the zero energy state. This takes care of the configurations with less than N dual-giants. The grand canonical partition function is, therefore,

$$\mathcal{Z}(\zeta, q_1, q_2, q_3) \equiv Tr \exp[-\mu N - \beta_i J_i] = \prod_{n_1=0}^{\infty} \prod_{n_2=0}^{\infty} \prod_{n_3=0}^{\infty} (1 - \zeta q_1^{n_1} q_2^{n_2} q_3^{n_3})^{-1} \tag{3.4}$$

The chemical potentials $q_i \equiv e^{-\beta_i}$ are conjugate to the charges J_i and the 'fugacity' $\zeta = e^{-\mu}$ is conjugate to number N of dual-giants:

$$\mathcal{Z}(\zeta, q_1, q_2, q_3) = \sum_{N=0}^{\infty} \zeta^N Z_N(q_1, q_2, q_3) \tag{3.5}$$

with

$$Z_N(q_1, q_2, q_3) = \sum_{J_1, J_2, J_3=0}^{\infty} \Omega_N(J_1, J_2, J_3) q_1^{J_1} q_2^{J_2} q_3^{J_3}. \tag{3.6}$$

Some special configurations however have enhanced supersymmetries to either 1/4 or half-BPS states and so we may choose to subtract them out. For this a useful way to think about this counting is the following. A generic state can be specified by N identical bosonic particles sitting on the 3-dimensional lattice with each lattice point coordinates n_1, n_2, n_3 which take integer values. The half-BPS states again are the configurations of points on this 3-dimensional lattice where all N points fall on a single line going through the origin. The 1/4-BPS states are those for which all N points fall on a single 2-plane going through the origin again. We will discuss counting of 1/4-BPS states separately in the next subsection.

3.2 $\frac{1}{4}$-BPS states

As stated above, dual-giant configurations for which all N dual-giants have the same specific α and β preserve 1/4 of the supersymmetries (eight supersymmetries). In terms of the bosons on a 3-dimensional lattice these are the configurations for which all bosons lie on the same 2-plane passing through the origin. The choice of which eight supersymmetries we want to preserve fixes the values of α and β (and therefore fixes a plane in the 3-d lattice). For the purpose of counting, we may choose this plane to be the $\mu_3 = 0$ plane ($\beta = \pi/2$), which in turn fixes $P_{\xi_3} = 0$. Classically,

$$P_{\xi_1} = \frac{Nr^2}{l^2} \sin^2 \alpha, \quad P_{\xi_2} = \frac{Nr^2}{l^2} \cos^2 \alpha. \tag{3.7}$$

Repeating the same steps as in section 2.4, we can now find a four-dimensional phase space with the symplectic structure of $\mathbb{C}^2$, expressed by the following Dirac brackets

$$\{\xi_1, r^2 \sin^2 \alpha\}_{DB} = \frac{l^2}{2N}, \quad \{\xi_2, r^2 \cos^2 \alpha\}_{DB} = \frac{l^2}{2N}. \tag{3.8}$$

As before, for fixed α they preserve 16 supersymmetries. But if we put together dual-giants with various values of α that configuration preserves only 8 supersymmetries.

Quantization proceeds in a manner similar to the 1/8-BPS case. The conserved momenta (P_{ξ_1}, P_{ξ_2}) are identified as the quantized angular momenta (J_1, J_2). Counting the 1/4-BPS states goes as follows. To make a 1/4-BPS state we need to have at least two dual-giants in that configuration with different values of α. The specific values of α should be such that we get integer values for J_1 and J_2. The other constraint is that one can have a total of no more than N dual-giant gravitons.

So the 1/4-BPS states can be counted by considering the number of ways, $\Omega_N(J_1, J_2)$, in which one can distribute N identical bosons on a 2-dimensional lattice of integers (n_1, n_2), such that the sum of their individual n_1's is J_1 and the sum of the individual n_2's is J_2. A partition function which generates this number is

$$\mathcal{Z}(\zeta, q_1, q_2) = \prod_{n_1,n_2=0}^{\infty} (1 - \zeta\, q_1^{n_1} q_2^{n_2})^{-1} = \sum_{N=0}^{\infty} Z_N(q_1, q_2)\zeta^N \tag{3.9}$$

The above counting includes states which preserve at least 1/4 of the supersymetries. To exclude the special states which preserve 1/2 of the supersymmetries one needs to exclude configurations in which all particles lie on a straight line passing through the origin. Each line passing through the origin in a 2-dimensional lattice can be uniquely specified by a pair of relatively prime integers, say $(n_1, n_2) = (r, s)$. This line (r, s) corresponds to the points $(n_1, n_2) = k\,(r, s)$, $k = 1, \ldots, \infty$. So to eliminate the contribution from the states with enhanced supersymmetries one should simply subtract the contribution of configurations in which all N particles lie on the same line from $Z_N(q_1, q_2)$ defined in (3.9).

3.3 Comparison with gauge theory answers

Single trace 1/4-BPS operators of $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group have been constructed in the literature [16 − 18]. These states belong to the $[p, q, p]$ representation of $SU(4)$ which has $(J_1, J_2) = (p + q, p)$. More generally, an index for $\mathcal{N} = 4$ Yang-Mills theories has recently been calculated [19] (see also [20]) which counts 1/8-, 1/4- and 1/2-BPS states of the kind we discussed above. Our results (3.4) and (3.9) agree with their result. In case of the 1/8-BPS states, the dual-giant graviton states we constructed above are to be identified with gauge-invariant operators which do not involve the fermionic fields.

4. 1/8-BPS states with (S_1, S_2, J_1)

In this section we consider a different counting problem, namely that of configurations of multiple giant gravitons which carry non-zero (S_1, S_2, J_1) charges and preserve at least 4 supersymmetries. Similar configurations have been considered in the literature before in [21, 22]. We work in the coordinates that we used in the earlier sections and consider configurations of D3-branes of the type:

$$\begin{array}{lllll} t = \tau, & ,r = r(\tau), & \theta = \theta(\tau), & \phi_1 = \phi_1(\tau), & \phi_2 = \phi_2(\tau), \\ \alpha = \alpha(\tau), & \beta = \sigma_1, & \xi_1 = \xi_1(\tau), & \xi_2 = \sigma_2, & \xi_3 = \sigma_3 \end{array} \tag{4.1}$$

The pull-back of the RR 4-form is $C^{(4)}_{\tau\sigma_1\sigma_2\sigma_3} = -l^4 \cos^4\alpha \sin\sigma_1 \cos\sigma_2\ \dot{\xi}_1$. Then the D3-brane Lagrangian becomes

$$L = -\frac{N \cos^3\alpha}{l}\left[\Delta^{1/2} - l\cos\alpha\,\dot{\xi}_1\right] \tag{4.2}$$

where

$$\Delta = V(r) - \frac{\dot{r}^2}{V(r)} - r^2\,(\dot{\theta}^2 + \cos^2\theta\,\dot{\phi}_1^2 + \sin^2\theta\,\dot{\phi}_2^2) - l^2(\dot{\alpha}^2 + \sin^2\alpha\,\dot{\xi}_1^2) \tag{4.3}$$

Notice the change of sign for the Chern-Simons term. The reason for this is we have chosen here an anti-D3-brane rather than a D3-brane so as to find a solution with positive sign of $\dot{\xi}_1$. It is easy to see that the following configurations satisfy the equations of motion

$$\dot{\alpha} = \dot{r} = \dot{\theta} = 0, \quad |\dot{\phi}_1| = |\dot{\phi}_2| = \dot{\xi}_1 = 1/l \tag{4.4}$$

In fact these configurations saturate a Bogomolny bound and the Hamiltonian reads (for positive values of $l\,\dot{\phi}_1$, $l\,\dot{\phi}_2$)

$$H = \frac{1}{l}(P_{\phi_1} + P_{\phi_2} + P_{\xi_1}) \tag{4.5}$$

where

$$P_{\xi_1} = N\,\cos^2\alpha, P_{\phi_1} = (r^2/l^2)\,P_{\xi_1}\,\cos^2\theta, P_{\phi_2} = (r^2/l^2)\,P_{\xi_1}\,\sin^2\theta. \tag{4.6}$$

We will show in the next subsection that the configurations (4.4) share at least 4 supersymmetries for arbitrary values of r, θ and α. In terms of the canonical variables, the supersymmetry conditions (4.4) imply the following constraints

$$\begin{array}{rrr} P_r = 0, & P_\theta = 0, & P_\alpha = 0, \\ P_{\phi_1} - \frac{r^2}{l^2}\,N\,\cos^2\alpha\,\cos^2\theta = 0, & P_{\phi_2} - \frac{r^2}{l^2}\,N\,\cos^2\alpha\,\sin^2\theta = 0, & P_{\xi_1} - N\,\cos^2\alpha = 0. \end{array} \tag{4.7}$$

As for the case of the dual-giant gravitons earlier, we can define

$$(r_1, r_2, r_3) = (r\,\cos\alpha\,\cos\theta, r\,\cos\alpha\,\sin\theta, l\,\cos\alpha)$$

and

$$\zeta_1 = r\,\cos\alpha\,\cos\theta\,e^{i\phi_1},\ \zeta_2 = r\,\cos\alpha\,\sin\theta\,e^{i\phi_2},\ \zeta_3 = l\,\cos\alpha\,e^{i\,\xi_1}$$

Because of the six constraints (4.7) the 12-dimensional phase space of the $\zeta_i, \bar{\zeta}_i$ and their conjugate momenta gets reduced to a six-dimensional phase space. The problem of finding the Dirac brackets in the reduced phase space is similar to the previous case and we get

$$\{\zeta_i, \bar{\zeta}_j\}_{DB} = -i\,\frac{l^2}{N}\,\delta_{ij} \tag{4.8}$$

which gives the same symplectic form, eq. (2.38), as before.

The main difference of this solution space from the earlier one is that $|\zeta_3| \leq l$. So the phase space is really $\mathbb{C}^2 \times D$ with the symplectic form (2.38) inherited from $\mathbb{C}^3$. Note that the boundary is a null curve of the symplectic form which is as it should be.

In terms of the new variables, (4.6) reads

$$P_{\phi_1} = \frac{N}{l^2}\,|\zeta_1|^2, \quad P_{\phi_2} = \frac{N}{l^2}\,|\zeta_2|^2, \quad P_{\xi_1} = \frac{N}{l^2}\,|\zeta_3|^2 \tag{4.9}$$

and the Hamiltonian written in these coordinates becomes

$$l\,H = \frac{N}{l^2}\sum_{i=1}^{3}\zeta_i\bar{\zeta}_i \tag{4.10}$$

Eqs. (4.10), (4.8) imply that the system is again a 3-dimensional simple harmonic oscillator (the implication of bounded $|\zeta_3|$ for quantization will be discussed shortly).

JHEP03(2007)031

4.1 Supersymmetries of spinning giants

The world-volume gamma matrices are

$$\gamma_\tau = V^{1/2}\,\Gamma_0 + \frac{\dot r}{V^{1/2}}\,\Gamma_1 + r\,\dot\theta\,\Gamma_2 + r\,(\dot\phi_1\,\Gamma_3\,\cos\theta + \dot\phi_2\,\Gamma_4\,\sin\theta) + l\,(\dot\alpha\,\Gamma_5 + \sin\alpha\,\dot\xi_1\,\Gamma_7),$$
$$\gamma_{\sigma_1} = l\,\cos\alpha\,\Gamma_6,\quad \gamma_{\sigma_2} = l\,\mu_2\,\dot\xi_2\,\Gamma_8,\quad \gamma_{\sigma_3} = l\,\mu_3\,\dot\xi_3\,\Gamma_9. \tag{4.11}$$

The kappa projection equations for an anti-D3-brane are

$$\Big[V^{1/2}\,\Gamma_0 + \frac{\dot r}{V^{1/2}}\,\Gamma_1 + r\,\dot\theta\,\Gamma_2 + r\,(\dot\phi_1\,\Gamma_3\,\cos\theta + \dot\phi_2\,\Gamma_4\,\sin\theta)$$
$$+l\,(\dot\alpha\,\Gamma_5 + \sin\alpha\,\dot\xi_1\,\Gamma_7) - i\,\Delta^{1/2}\,\Gamma_{689}\Big]\epsilon = 0 \tag{4.12}$$

On the solutions $\dot r = 0$, $\dot\theta = 0$, $\dot\alpha = 0$, $\dot\phi_1 = \dot\phi_2 = \dot\xi_1 = 1/l$ this reduces to

$$\Big[V^{1/2}\,\Gamma_0 + \frac{r}{l}(\Gamma_3\,\cos\theta + \Gamma_4\,\sin\theta) + \Gamma_7\,\sin\alpha - i\Gamma_{689}\,\cos\alpha\Big]\,\epsilon = 0. \tag{4.13}$$

To simplify this equation we use the following identity

$$(\sin\alpha\,\Gamma_7 - i\,\cos\alpha\,\Gamma_{689})M = -i\,M\,\Gamma_{57}\tilde\gamma \tag{4.14}$$

Then the killing spinor equation can be rewritten as

$$M[V\,\Gamma_0 + \frac{r}{l}V^{1/2}\,\cos\theta\,e^{-\phi_1\,\Gamma_{13} - i\frac{t}{l}\Gamma_0\,\gamma}(\Gamma_3 + i\,\Gamma_{01}\,\gamma)$$
$$+\frac{r}{l}V^{1/2}\,\sin\theta\,e^{-\phi_1\,\Gamma_{24} - i\frac{t}{l}\Gamma_0\,\gamma}(\Gamma_4 + i\,\Gamma_{02}\,\gamma) - i\frac{r^2}{l^2}(\cos^2\theta\,\Gamma_{13} + \sin^2\theta\,\Gamma_{24})\gamma$$
$$-i\,\frac{r^2}{l^2}\cos\theta\,\sin\theta\,(\Gamma_{23} + \Gamma_{14})\,\gamma\,e^{-\phi_1\,\Gamma_{13} - \phi_2\,\Gamma_{24}} - i\,\Gamma_{57}\,\tilde\gamma]\,\epsilon_0 = 0. \tag{4.15}$$

Since we are interested in having configurations of multiple giant gravitons with generically different values of (r,θ,α) we may impose the projections

$$(\Gamma_0 - i\,\Gamma_{13}\gamma)\,\epsilon_0 = 0,\quad (\Gamma_0 - i\,\Gamma_{24}\gamma)\,\epsilon_0 = 0,\quad (\Gamma_0 - i\,\Gamma_{57}\,\tilde\gamma)\,\epsilon_0 = 0. \tag{4.16}$$

with which the killing spinor equation is satisfied. So the solutions we have found share at least 4 supersymmetries.

Using methods similar to the ones used in section (2.3) one may show that the full set of solutions given the projections in (4.16) is the one considered above.

Further one can interpret the solutions as all those obtained by $U(1,2)$ rotations acting on the homomorphic coordinates (Φ_0, Φ_1, Φ_2) of $\mathbb{C}^{1,2}$ (in which AdS_5 is embedded as $-|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 = -l^2$) on the original half-BPS giant graviton.

4.2 Quantization

The phase space is $\mathbb{C}^2 \times D$ where D represents the Disc $|\zeta_3| \leq 1$.

The D part of the phase space (with the symplectic form and the Hamiltonian) is identical to that of the half-BPS giant gravitons [15]. The semiclassical quantization corresponds to a fuzzy disc and the exact single-particle quantum mechanics is described by

$H_{1,N}$ which is an N-dimensional Hilbert space consisting of the first N levels of a simple harmonic oscillator [15].

$$\mathcal{H}^{1,N} = \mathrm{Span}\{|n\rangle, n = 1, 2, \ldots, N\} \tag{4.17}$$

We have omitted the $n = 0$ state from the spectrum of giant gravitons since the minimum angular momentum of a half-BPS giant graviton is unity. The single-particle quantum mechanics corresponding to $\mathbb{C}^2$ is given by the states of a 2-dimensional simple harmonic oscillator of arbitrarily high quantum numbers:

$$\mathcal{H}' = \mathrm{Span}\{|l, m\rangle, l, m = 0, 1, \ldots, \infty\} \tag{4.18}$$

The full single-particle Hilbert space of the giant gravitons is given by $\mathcal{H}_1 = \mathcal{H}^{1,N} \times \mathcal{H}'$.

The three angular momenta (4.6) are identified now as the three charges (S_1, S_2, J_1) where the first two correspond to angular momenta in AdS_5 and J_1 correspond to angular momentum in S^5.

$$Z(q_1, q_2, q_3) = \prod_{l,m=0}^{\infty} \prod_{n=1}^{N} \frac{1}{1 - q_1^l\, q_2^m\, q_3^n} = \sum_{S_1,S_2,J_1=0}^{\infty} \Omega_N(S_1, S_2, J_1)\, q_1^{S_1}\, q_2^{S_2}\, q_3^{J_1} \tag{4.19}$$

Similar to the situation in the 1/8-BPS states with (J_1, J_2, J_3) one has configurations in this partition function which have more than 4 supersymmetries. For example whenever there is just one giant graviton making up a state, since it can be obtained by an isometry of AdS_5 acting on the standard half-BPS giant graviton it is expected to preserve 16 supercharges [22] (see also [21]). Further whenever all the giants in a given state have the same value of r that state is expected to preserve 8 supercharges. Again if one wishes one can systematically exclude the contribution of these states from (4.19) to get the degeneracies of exactly 1/8 or 1/4 supersymmetric states.

5. Conclusions

In this paper we considered the counting problem of quantum states in the type IIB string theory on $AdS_5 \times S^5$ background that preserve at least 4 supercharges (1/8-BPS). Two types of states have been considered:

- those with non-zero (J_1, J_2, J_3) with $E = J_1 + J_2 + J_3$ where E is their energy conjugate to the time coordinate in global coordinates and J_i are the three independent angular momenta on S^5,
- those with non-zero (S_1, S_2, J_1) with $E = S_1 + S_2 + J_1$ where S_i are the two independent angular momenta on $S^3 \subset AdS_5$.

For the first set we considered N-particle states of dual-giant gravitons rotating along arbitrary maximal circles of S^5 that share at-least four supersymmetries. The result can be expressed quite simply in terms of the degeneracy of states of an N-boson system in a three-dimensional harmonic oscillator potential.

JHEP03(2007)031

For the second set of 1/8-BPS states we considered configurations containing an arbitrary numbers of giant gravitons having angular momenta in the AdS_5 directions which share at least 4 supersymmetries. The result for the counting problem for this set of states can again be mapped onto that of a 3-dimensional harmonic oscillator, but this time with an arbitrary number of bosons representing the giant gravitons and with one of the three quantum numbers of the 3-dimensional harmonic oscillator bounded above by N.

The first type of 1/8-BPS states can also be given by the classification of giant gravitons of [11]. Recently [12] have counted these 1/8-BPS giant graviton configurations by quantizing Mikhailov's solutions (See also [28]). The counting problem in terms of giant gravitons appears to be significantly more complicated since the giant gravitons wrap different 3-surfaces within the S^5 which can have complicated intersections and quantization involves quantizing the space of these 3-surfaces. The dual-giant gravitons, however, have a much simpler description since their world-volume, at any given time, is a three-sphere of a given size inside AdS_5 and their motion is that of point particles on the S^5. Remarkably, both these descriptions give the same counting of 1/8-BPS states. This points to some duality between the giant graviton and the dual-giant graviton descriptions.

It is pertinent to recall at this point [6, 15] that in case of half-BPS states there is an explicit duality between giant graviton states and dual-giant graviton states. If we specify a multi-giant graviton state by $(r_1, r_2, \cdots, r_N)$ where r_i denotes the number of giant gravitons with angular momentum $J_1 = i$ and a multi-dual-giant graviton state by $(s_1, s_2, \cdots, s_N)$ where s_i is the angular momentum of the i$^{\text{th}}$ (counting from the dual-giant with largest J_1) then the duality map is given by

$$s_i = \sum_{k=i} r_k \tag{5.1}$$

Thus, the number of dual-giants becomes the number of single-particle energy levels for giants. Also, the occupied energy levels (s_i) of dual-giants get related (5.1) to occupation numbers (r_k) of individual levels in the giant graviton system. It will be very interesting to explore if this type of a duality exists for the 1/8-BPS states with (J_1, J_2, J_3) quantum numbers considered here and in [11, 12]. Further in the half-BPS sector the duality between giants and dual-giants follows form a unified description of the system [23, 24] in terms of N fermions in a harmonic oscillator potential. It will be interesting to see if such a unified picture can be given for lower supersymmetric cases too. It is possible that such a description arises in the solution of matrix models proposed in [25] to capture the physics of 1/8-BPS states.

Similar to the description of giant gravitons in [11] one can describe a dual-giant graviton as a three surface obtained by the intersection of $\Phi_0 = \varphi_0^{(0)}$ and $Z_1 = \zeta_1^{(0)}$, $Z_2 = Z_3 = 0$ with $\mathbb{C}^{1,2} \times \mathbb{C}^3$ and $-|\Phi_0|^2 + |\Phi_1|^2 + |\Phi_2|^2 = -l^2$ and $\sum_{i=1}^{3} |Z_i|^2 = l^2$ evolving in time as $\Phi_0 \to \Phi_0\, e^{i\frac{t}{l}}$ and $Z_1 \to Z_1\, e^{i\frac{t}{l}}$ where (Φ_0, Φ_1, Φ_2) are holomorphic coordinates on $\mathbb{C}^{1,2}$ and (Z_1, Z_2, Z_3) are holomorphic coordinates on $\mathbb{C}^3$ and $\varphi_0^{(0)}$, $\zeta_1^{(0)}$ are arbitrary complex numbers. This can be generalized to 'wobbling dual-giants' [26]

$$g(\Phi_0, \Phi_1, \Phi_2) = 0, \quad Z_1 = \zeta_1^{(0)}, \quad Z_2 = Z_3 = 0 \tag{5.2}$$

with the time evolution $\Phi_i \rightarrow \Phi_i e^{i\frac{t}{l}}$ and $Z_1 \rightarrow Z_1 e^{i\frac{t}{l}}$. These states carry non-zero (S_1, S_2, J_1) generically and can be shown to preserve at least 1/8 of the supersymmetries of the background [26]. One should be able to quantize the space of these 'wobbling dual-giants' and count the 1/8-BPS states using the methods of [12]. It will be interesting to see if there exists a duality between our giant graviton configurations and these wobbling dual-giants in which case our counting results give a prediction for the dual-giant counting.

In both types of 1/8-BPS states we considered here preserve $SO(4)$ symmetry (coming from isometries of $S^3 \subset AdS_5$ for the states with (J_1, J_2, J_3) charges and from $S^3 \subset S^5$ for those with (S_1, S_2, J_1) charges). So different giants in a given state form concentric three-spheres and never intersect (and whenever they do they actually coincide). Usually D-branes which intersect are expected to split and rejoin and therefore the degeneracies of such states can change. But in our case since they do not intersect we suggest that their degeneracies should not receive any quantum corrections. It will be interesting to see if one can give a description of the full set of 1/8-BPS states with given charges in our language by turning on some bosonic or fermionic zero modes on the world-volume of the probe branes that break the $SO(4)$ invariance but not supersymmetry.

It is of interest to find an exact orthonormal basis of states in the dual $\mathcal{N} = 4$ $U(N)$ SYM for the 1/8-BPS states considered here. For half-BPS states an orthonormal basis of operators in SYM was provided in [27]. See [22] for some comments on the dual operators for the states with non-zero (S_1, S_2, J_1).

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